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DEPTH FORMULA AND VANISHING OF (CO)HOMOLOGY

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Abstract. In the first part of this talk, we will discuss about the depth formula over Gorenstein rings, which was initially introduced by Auslander over regular local rings. We will show that the depth formula holds for nonzero finitely generated modules M and N in case certain Gorenstein relative and Tate homology modules vanish. In the second part, we will talk about Auslander–Reiten Conjecture. We will present various criteria for freeness of modules over local rings in terms of vanishing of Ext modules, which recover a lot of known results on the Auslander–Reiten Conjecture.

1. Introduction

For finitely generated modules M and N over a commutative Noetherian local ring R, the pair (M, N) is said to satisfy the *depth formula* provided

$$\operatorname{depth} R + \operatorname{depth}_R(M \otimes_R N) = \operatorname{depth}_R(M) + \operatorname{depth}_R(N).$$

The depth formula was first studied by Auslander. In [2], Auslander proved that the pair (M, N) satisfies the depth formula provided

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that the projective dimension of M is finite and M and N are Tor-independent, i.e., $\operatorname{Tor}_i^R(M,N)=0$ for all $i>0$. Three decades later Huneke and Wiegand proved that the depth formula holds for Tor-independent modules over complete intersection local rings. Araya and Yoshino, and Iyengar independently generalized Auslander's result for modules of finite complete intersection dimension. The finite complete intersection dimension condition was then relaxed by Christensen and Jorgensen [4]. More precisely, they proved the following result:

Theorem 1.1 (Christensen and Jorgensen)**.** *Let M and N be R–modules such that M has finite Gorenstein dimension. Then the depth formula holds provided the following conditions hold.*

(1) $\operatorname{Tor}_i^R(M,N)=0$ *for all* $i>0$.
(2) $\widehat{\operatorname{Tor}}_i^R(M,N)=0$ *for all* $i\in\mathbb{Z}$.

In the first part of this talk, we will discuss about the depth formula over Gorenstein rings. We will show that the depth formula holds for nonzero finitely generated modules M and N in case certain Gorenstein relative and Tate homology modules vanish.

In the second part, we will talk about Auslander–Reiten Conjecture. This is one of the most celebrated conjectures in the representation theory of algebras. This long-standing conjecture is known to hold true over several classes of algebras, including algebras of finite representation type and symmetric artin algebras with radical cube zero. This conjecture is closely related to other conjectures such as the *Nakayama Conjecture* and the *Tachikawa Conjecture*. Although the Auslander–Reiten Conjecture was initially proposed over artin algebras, it remains meaningful for arbitrary commutative noetherian rings:

Conjecture 1.2 (Auslander–Reiten)**.** *Let R be a commutative noetherian ring R and let M be a finitely generated R-module. If $\operatorname{Ext}_R^i(M,M)=0=\operatorname{Ext}_R^i(M,R)$ for all $i\geq 1$, then M is projective.*

Auslander, Ding and Solberg proved that Conjecture 1.2 holds for any complete intersection local rings. Recently there has been various progress towards Conjecture 1.2. Using Auslander–Reiten duality, Araya [1] proved that if all Gorenstein local rings of dimension at most one satisfy the Auslander–Reiten Conjecture, then so do all Gorenstein local rings. We will present various criteria for freeness of modules over local rings in terms of vanishing of Ext modules, which recover a lot of known results on the Auslander–Reiten Conjecture.

2. Main sections and results

Recently, in a joint work with Celikbas and Liang [3], we obtain a new condition that is sufficient for the depth formula to hold. A new tool we use is the G-relative homology $\mathcal{G}\mathrm{Tor}_*^R(M,N)$ which has been defined and studied by Avramov and Martsinkovsky. We establish that the depth formula holds under weaker assumptions:

Theorem 2.1. *(Celikbas–Liang–Sadeghi) Let M and N be R–modules such that M has finite Gorenstein dimension. Then the depth formula holds provided the following conditions hold.*

(1) $\mathcal{G}\mathrm{Tor}_i^R(M,N)=0$ *for all* $i>0$.
(2) $\widehat{\mathrm{Tor}}_i^R(M,N)=0$ *for all* $i\leq 0$.

Let us remark that the hypotheses of Theorem 2.1 are in general weaker than those of Theorem 1.1: Jorgensen and Şega constructed modules M and N with $\widehat{\mathrm{Tor}}_i^R(M,N)=0$ for all $i\leq 0$, $\mathcal{G}\mathrm{Tor}_i^R(M,N)=0$ for all $i\geq 1$ and $\widehat{\mathrm{Tor}}_i^R(M,N)\neq 0$ for all $i\geq 2$. Moreover relative homology vanishes more frequently than absolute homolgy. For example, if M is totally reflexive, i.e., M has $\mathcal{G}$-dimension zero, then $\mathcal{G}\mathrm{Tor}_i^R(M,N)=0$ for all $i\geq 1$. In particular this establishes, if M is totally reflexive and $\widehat{\mathrm{Tor}}_i^R(M,N)=0$ for all $i\leq 0$, then the depth formula holds.

In the second part, we will talk about Auslander–Reiten Conjecture. Recently there has been various progress towards this Conjecture. A particular result worth recording on the Auslander–Reiten Conjecture is due to Huneke and Leuschke: the Auslander–Reiten Conjecture holds over Gorenstein normal domains. In a joint work with Takahashi [5], we generalize the Huneke–Leuschke's result as follows:

Theorem 2.2. *(Sadeghi–Takahashi) Let R be a noetherian normal local ring of depth t. A finitely generated R-module M is free, if either of the following conditions holds.*

(1) $\mathrm{Ext}_R^i(M,M)=\mathrm{Ext}_R^j(M,R)=\mathrm{Ext}_R^h(\mathrm{Hom}_R(M,M),R)=0$ *for all* $1\leq i\leq t-1$, $j\geq 1$ *and* $2\leq h\leq t$.
(2) $\mathrm{Ext}_R^i(M,M)=\mathrm{Ext}_R^j(M,R)=0$ *for all* $1\leq i\leq t-1$ *and* $j\geq 1$, *and that* $\mathrm{Hom}_R(M,M)$ *has finite* G*-dimension.*

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