Representations and Characters of Finite Groups: Old and New Results

M. R. Darafsheh
School of Mathematics, Statistics, and Computer Science,
College of Science, University of Tehran, Tehran, Iran.
e-mail : $darafsheh@ut.ac.ir$

Abstract

The recent progress in representation and character theory of finite groups is investigated. In this respect results by many group theorists are mentioned.

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1 Introduction

Let G be a finite group. A representation of G is a homomrphism $T : G \to GL(V, F)$ where V is a finite-dimesioned vector space over the field F, called the representation space of G. $GL(V, F)$ is the group of all the invertible linear transformations of V. If $dim(V) = n$, then $GL(V, F) \cong GL_n(F)$ the group of all $n \times n$ invertible matrices over F. Therefore we may think of T as a homomorphism $T : G \to GL(V, F)$, $T(1) = I_n$ is called the degree of the representation.

T is called faithful if $kerT = 1$. In this case G is isomorphic to a subgroup of $GL_n(F)$. If $T(g) = I_n$ for all $g \in G$ then T is called a trivial representation of G. If $Char(F) = p||G|$ then T is called a modular representation of G, in case of $Char(F) = 0$, T is called an ordinary representation of G.

In this talk we are concerned with ordinary representations, in particular we assume $F = \mathbb{C}$, the field of complex numbers.
If $T : G \to GL(V, F)$ is a representation of G, then by defining: $g(V) := T(G)v, \forall g \in G$; $\forall v \in V$ then V becomes an FG-module.
Conversely if V is an FG-module, then with definition: $T(G)v := g(v), \forall g \in G; \forall v \in V$, is a representation of G.
There is a $1-1$ correspondence between representations of G and FG-modules. T is called reducible if there is a non-zero proper submodule U of V such that $T(g)U \subseteq U$. Otherwise T is called irreducible.
Two representations $S : G \to GL(V, F)$ and $T : G \to GL(U, F)$ are called equivalent if V and U as FG-modules are isomorphic.
The number of non-equivalet irreducible representation of G is equal to the number of conjugacy classes of G.

Theorem 1.1. *(Artin-Wedderburn) If G has k conjugacy classes, then:*

$$FG \cong Mat_{n_1}(F) \bigoplus ... \bigoplus Mat_{n_k}(F)$$

$n_1, ...n_k$ *are the degrees of non-equivalent representations of G.*

To each representation $T : G \to GL_n(F)$ we associate the function $\chi : G \to F$ by $\chi(g) = trT(g)$, $\forall g \in G$ and call it the character afforded by T.
$\chi(g)$ is an algebraic integer.
$\chi(1) = trT(1) = n$, the degrees of χ.
If $\chi(1) = 1$ then χ is called a linear character of G.
The character χ is called reducible or irreducible if the representation T affording T is reducible or irreducible.
All concepts about characters are the same as for representations. Equivalent representations have the same characters. Each character is constant member of a conjugacy class. The number of irreducible characters of G is equal to the number of conjugacy classes of G. Characters may be added or multiplied:

$$(\chi + \varphi)(g) = \chi(g) + \varphi(g)$$

$$(\chi\varphi)(g) = \chi(g)\varphi(g)$$

$Irr(G)$ is the set of all the irreducible characters of G.

$cd(G)$ is the set of degrees of irreducible characters of G.

$Lin(G)$ is the set of linear characters of G.

Irreducible characters of G satisfying orthogonality relations:

If χ is an irreducible character of G, then $\Sigma_{g\in G}\chi(g)\chi\bar{(g)} = |G|$ and if $\varphi \neq \chi$ is an irreducible character of G, then $\Sigma_{g\in G}\chi(g)\chi\bar{(g)} = 0$.

Let $\chi_1, ..., \chi_h$ be all the irreducible characters of G and $C_1, ..., C_h$ be all the conjugacy classes of G and $g_i \in C_i, 1 \leq i \leq h$. Let us form the table:

	$g_1 = 1$	g_2	.	.	g_j	.	.	g_h	
χ_1					.				
.					.				
.					.				
.					.				
χ_i	.	.	.	.	$\chi_i(g_j)$	.	.	.	.
.					.				
.					.				
.					.				
χ_h					.				

The above table is called the character table of G. Many information about G can be derived from the character table. Some classical theorems are proved with the help of characters Burnside's $p^a q^b$-theorem:

Every groups of order $p^a q^b$ is solvable, p, q prime numbers.

Frobenius kernel: Let G be a transitive permutation group on Ω such that $G_\alpha \neq 1$ but $G_{\alpha,\beta} = 1$ for all $\alpha, \beta \in \Omega$. Then the set K of fixed-point-free elements of G together with the identity forms a normal subgroup of G.

2 Computation of character table

Representation and character theory of finite groups was invented by G. F. Frobenius in 1896.

The first book on finite groups was written by W.Burnside in 1897, who wrote important papers on characters of finite groups in 1900.

Schur has obtained results on characters of symmetric groups from 1904-

R. Brauer has results on p-modular characters and representation of finite groups,1935-

Informtion about the character tables of simole groups and Their automorphism groups is compiled in:

An atlas of finite groups by H. H. Conway, R. Curtis, S. P. Norton, P. A. Parker and R. A. Wilson, Oxford university press, 1985.

Character tables of the general linear groups:

[R. Steinberg, The representations of $GL_3(q), GL_4(q), PGL_3(q), PGL_4(q)$, Canad J. Math. Vol.3, 1951, 225-235]

3 Character tables of simple groups not appearing in ATLAS

[M. R. Darafsheh, computing the irreducible characters of the group $GL_6(2)$, Math. comput. , vol46, no. 173(1986), 301-319]

[M. R. Darafsheh, characters of the automorphism group of the group $GL_6(2)$, J. Alg. , vol. 108, No. 1, 1987, 256-268]

character table of $GL_7(2)$ computed by M. Khademi, 1993.

character table of $\mathbb{A}ut(GL_7(2))$ computed by Z. Mostaghim,1997.

character table of $GL_8(2)$ computed by A. Daneshkhah, 1998.

character table of $SL_5(3)$ computed by M. R. Tarkhorani, 1993.

character table of $\mathbb{A}ut(SL_5(3))$computed by A. Ashrafi, 1995.

In some general cases the character tables of groups of Lie type are investigated. We mention a few:

[J. A. Green, the characters of the finite general linear groups, Trans. Amer . Math . Soc. 80(1995), 402-447]

[B. Srinivasan, The characters of the finite symplectic groups $SP_4(q)$, Trans. Amer. Math. Soc.131(1968), 488-525]

4 Steinberg character

For a simple Lie type group $G(q)$ defined over a finite field $GF(q)$, q a power of the prime p, Steinberg proved that $G(q)$ has an irreducible character whose degree equals the order of a Sylow p-subgroup of $G(q)$ this character is called the Steinberg character of $G(q)$.
[R. Steinberg, Endomorphisms of linear algebraic groups, Memoirs of AMS, 80,(1968)]
Let G be a finite group and p be a prime number. A p-Steinberg character of G is an irreducible character χ of G such that $\chi(g) = \pm|C_G(x)|_p$ for every $p\prime$-element x in G and $\chi(x) = 0$ for every p-singular element x in G. It is easy to check that the Steinberg character of a simple group of Lie type is a p-Steinberg character.
Conjecture by W. Feit: G finite Simple group of order divisible by prime p. Suppose G has a p-Steinberg character, then G is a simple group of Lie type in characteristic p. This conjecture was proved in the following paper for the alternating and the projective special linear groups:
[M. R. Darafsheh, p-Steinberg character of alternating and projective special linear groups, J. Alg. , 191(1996), 196-206]
The conjecture was further investigated by P. H. Tiep and was proved using the classification of finite simple groups.
[P. H. Tiep, p-Steinberg characters of finite simple groups, J. Alg. vol. 187, Issue 1(1997) 304-319]

5 Characters of the affine classical groups

Consider the action of $GL_{n+1}(q)$ on $V_{n+1}(q)$, the stablilizer of a non-zero vector is a group of the form $A(n) = V_n(q)$. $GL_n(q)$ called the general affine group in dimension n. The additive group of $V_n(q)$ is isomorphic to an elementary abelian group, so A_n can be written as $q^n GL_n(q)$.

The group $Sp(2n+2,q)$ acts transitively on the vector space $V_{2n+2}(q)$ and stabilizer of a non-zero vector is isomorphic to $A(n) = q^{2n+1}Sp(2n,q)$ called affine symplectic group where q^{2n+1} is a p-group not necessarily abelian.

Similarly affine unitary group: $q^{2n-1}.U(n,q^2)$.

Similarly affine othogonal group: $q^{2n-1}.GO(2n-1,q)$.

The degrees of above irreducible characters of groups in terms of the degrees of the classical groups are obtained in a series of papers:

[M. R. Darafsheh, M. R. Tarkhorani and A. Daneshkhah, Irreducible complex characters of the full affine group, Int. J. Alg and computation vol. 5, No. 1 (1995) 1-5]

If $G(q)$ denotes one of the classical groups, then $A_n(q)$ has a unique character of degree $[G(q) : P]$, where P is a sylow p-subgroup of $G(q)$.

proof is by a method called Clifford-Fischer method which involves finding certain matrices called Fischer matrices.

6 Quasi-permutation representation

A square matrix over the complex field with non-negative integral trace is called a quasi-permutation matrix. For a finite group G, the minimal degree of a faithful representation of G is denoted by $p(G)$.

$q(G) =$ The minimal degree of a faithful representation of G by quasi-permutation matrices over $\mathbb{Q}$.

$c(G) =$ The same as above but over $\mathbb{C}$

$r(G) =$ Minimal degree of a faithful rational valued complex character of G.

Above quantities are found for different group:

[M. R. Darafsheh, et. al: Quasi-permutation representation of the group $GL_2(q)$, J. alg. 243(2001),142-167]

[M. R. Darafsheh and M. Ghorbani, Special representations of the group $SP_4(q)$ with minimal degrees, Acta Math. Hungar. , 102(2004) 287-296]

7 Huppert's conjucture

This conjcture was put forward in the following paper:

[B. Huppert, some simple groups which one determined by the set of their character degrees I, Ill. J. Math, 44(2000) 828-842]

Conjecture: Let G be a finite simple non-abelian group. If H is a finite group and $cd(H) = cd(G)$, then $H \cong A \times G$ where A is an abelian group.

The conjecture can be verified using CFSG. The conjecture is verified for most of sporadic simple groups as well as a few of simple groups of Lie type We can refer to the papers by H. P. Tong-Viet, T. P. Wakefield, C. Bessenrodt, H. Alavi and A. Daneshkhah.

[Huppert's conjecture for Fi_{23}, Rend. Sem. Math Univ padova, vol. 126(2011) 201-211]

8 Rational property of irreducible characters

If χ is an irreducible complex character of G, then $\chi(x)$ for all $x \in G$ is an algebraic integer.

$\mathbb{Q}(x)$ denotes the field generated by $\mathbb{Q}$ and all $x \in G$, if $\mathbb{Q}(x) = \mathbb{Q}$, then χ is called a rational character. G is called a rational group or a $\mathbb{Q}$-group if all of its irreducible characters are rational.

Examples are the Weyl group of the complex Lie algebras, some results:

If G is a solvabe $\mathbb{Q}$-group, then $\pi(G) \subseteq \{2, 3, 5\}$.

The only non-abelian $\mathbb{Q}$-groups are $SP_6(2)$ and $Q_8^+(2)$.

Non abelian composition factors of $\mathbb{Q}$-groups can only be:

$PSP_4(3), SP_6(2), O_8^+(2), PSL_3(4), PSU_4(3)$ or $\mathbb{A}_n, n \geq 5$.

G is a $\mathbb{Q}$-group if and only if x and x^m, $(m, o(x)) = 1$, are conjugate for all $x \in G$. In particular $\dfrac{N_G(< x >)}{C_G(< x >)} \cong \mathbb{A}ut(< x >)$. Frobenius $\mathbb{Q}$-groups:

(1) $E(3^n) : \mathbb{Z}_2, n \geq 1$.

(2) $E(3^{2m}) : \mathbb{Q}_8, m \geq 1$.

(3) $E(5^2) : \mathbb{Q}_8$.

[M. R. Darafsheh and H. Sharifi, Frobenius $\mathbb{Q}$-groups Arch. Math 83(2004) 102-105]

2-Frobinius $\mathbb{Q}$-groups:

G is called 2-Frobenius if there is a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\frac{G}{H}$ and K are Frobenius groups with kernel $\frac{K}{H}$ and H respectively.

Theorem 8.1. *If G is a Frobenius $\mathbb{Q}$-group, then G has a normal 2-subgroup N such that $\frac{G}{N} \cong \mathbb{S}_4$.*

[M. R. Darafsheh, A. Iranmanesh and S. A. moosavi, 2-Frobenius $\mathbb{Q} - groups$, Indian J. pure appl. Math. , 40(1)(2009), 29-34]

Long standing Conjecture: If G is a $\mathbb{Q}$-group then its Slow 2-subgroup is also a $\mathbb{Q}$-group.

In 2012 a counter example to the conjecture was found:

[I. M. Isaacs and G. Navarro, Sylow 2-subgroups of rational Solvable groups, Mat. Z. 272(2012), no.3-4, 937-945]

The counter example is a group of order $1536 = 2^9.3$ whose Sylow 2-subgroups has nilpotency class 3.

Generalizations of the concept of $\mathbb{Q}$-group:

G is called a $\mathbb{Q}_1$-group if all of its non-linear irreducible characters are rational valued. Every abelian group and $\mathbb{A}_4$ are examples of $\mathbb{Q}_1$-group.

G Frobenius $\mathbb{Q}_1$-group:

(1) $G \cong E(p^n) : \mathbb{Z}_t$, p odd prime, $t \geq 1$ even.

(2) $G \cong G\prime : \mathbb{Z}_t$, G' rational 2-group, $t \geq$ odd.

(3) $G \cong E(5^2) : \mathbb{Q}_8$ or $G \cong E(3^{2m}) : \mathbb{Q}_8$.

(4) $G \cong E(p^n) : H$, p Fermat prime, H metacyclic of order $2^m q$, q Fermat prime.

[M. Norooz-abelian and H. Sharifi, Frobenius $\mathbb{Q}_1$-group, Arch. Math. 105(2015), 509-517]

Another generalization:

$x \in G$ is said to be semi-rational if there exists a positive integer m such that every generator of $< x >$ is conjugate in G to either x or x^m. And x is called inverse semi-rational if every generator of $< x >$ is conjugate in G to x or x^{-1}.

If G is a sovable semi-rational, then $\pi(G) = \{2, 3, 5, 7, 13, 17\}$.

[D. chillag and S. Dolfi, semi-rational solvable groups, J. group theory, 13(4)(2010), 535-548]

Semi-rational Frobenius groups are investigated by M. R. Darafsheh, H. Alavi and A. Daneshkhah and semi-rational simple groups are investigated by H. Alavi and A. Daneshkhah:

[H. Alavi, A. Daneshkhah and M. R. Darafsheh, On semi-rational Frobenius groups, J. Alg. Appl; Vol.15, no. 2(2016)]

[H. Alavi and A. Daneshkhah, On semi-rational finite simple groups, Monatsh. Math.]