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T-CLIQUE IDEAL OF A GRAPH

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ABSTRACT. Given a finite simple graph G and a positive integer t, we associate a monomial ideal whose minimal generators correspond to t-cliques of G and call it the t-clique ideal of G. It generalizes the edge ideal notion in some sense, since the edge ideal of G can be seen as a 2-clique ideal. Our aim is to understand the algebraic properties of this ideal and generalize some results on edge ideals to t-clique ideals. A family of clique ideals with linear resolutions has been characterized. Also some families of graphs for which the quotient ring of their clique ideal is Cohen-Macaulay are introduced. Moreover, some algebraic properties and homological invariants of the clique ideal and its Alexander dual are studied for some special families of graphs. In the sequel, we specially consider the t-clique ideal of the complement of the path graph P_n. The set of associated primes of all powers of this ideal are described explicitly. It turns out that any such ideal has the persistence property. Also the index of stability and the stable set of associated prime ideals of this ideal are determined.

1. Introduction

Finding a correspondence between some families of squarefree monomial ideals and some combinatorial objects such as graphs and simplicial complexes and characterizing the algebraic invariants of the ideal

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in terms of the construction of the combinatorial object associated to it, has been studied extensively in the last few years. As the first sample of these ideals, squarefree monomial ideals of degree two had been considered as the edge ideals of simple graphs, which was first defined in [4]. Later, some other squarefree monomial ideals attached to graphs like path ideals, generalized cover ideals, et cetera, have been studied and some new families of ideals with special algebraic properties had been characterized. Classifying all monomial ideals with some algebraic properties like having a linear resolution or being Cohen-Macaulay in general is not easy to deal with. In this regard finding classes of monomial ideals with some special algebraic properties is important.

We introduce and study the t-clique ideal of a graph. For a graph G, a complete subgraph of G with t vertices is called a t-clique of G. The ideal $K_t(G)$ generated by the monomials $x_{i_1} \cdots x_{i_t}$ of degree t such that the induced subgraph of G on the set $\{x_{i_1}, \ldots, x_{i_t}\}$ is a complete graph, is called the **t-clique ideal** of G. Note that $K_2(G) = I(G)$.

We recall some preliminaries which are needed in the sequel. A simplicial complex Δ is called **shellable** if there exists an ordering $F_1 < \cdots < F_m$ on the facets of Δ such that for any $i < j$, there exists a vertex $v \in F_j \setminus F_i$ and $\ell < j$ with $F_j \setminus F_\ell = \{v\}$.

A vertex splittable ideal was defined in [2] as follows.

Definition 1.1. A monomial ideal I in $R = k[X]$ is called **vertex splittable** if it can be obtained by the following recursive procedure.

(i) If u is a monomial and $I = \langle u \rangle$, $I = \langle 0 \rangle$ or $I = R$, then I is a vertex splittable ideal.

(ii) If there is a variable $x \in X$ and vertex splittable ideals I_1 and I_2 of $k[X \setminus \{x\}]$ so that $I = xI_1 + I_2$, $I_2 \subseteq I_1$ and $\mathcal{G}(I)$ is the disjoint union of $\mathcal{G}(xI_1)$ and $\mathcal{G}(I_2)$, then I is a vertex splittable ideal.

A graph G is called **chordal**, if it contains no induced cycle of length greater than or equal to 4. Also G is called **co-chordal** if the complement graph G^c is a chordal graph. A path graph with n vertices is denoted by P_n and a cycle graph with n vertices is denoted by C_n.

Throughout this manuscript, we assume that G is a simple graph with the vertex set $V(G) = \{x_1, \ldots, x_n\}$ and $R = k[x_1, \ldots, x_n]$ is a polynomial ring over a field k.

2. Algebraic Properties of t-clique ideals

In this section we study some algebraic properties of t-clique ideal of some families of graphs like co-chordal graphs and the complements of path graphs and cycle graphs.

Theorem 2.1. *Let G be a chordal graph such that $K_t(G^c) \neq 0$. Then $K_t(G^c)$ is a vertex splittable ideal for any positive integer t. Hence it has linear quotients and a t-linear resolution.*

Corollary 2.2. *Let G be a chordal graph and u be a simplicial vertex of G. Set $I = K_t(G^c)$, $J = K_t((G \setminus u)^c)$ and $K = K_{t-1}((G \setminus N_G[u])^c)$. Then $I = J + uK$ is a Betti splitting. Moreover, if $I \neq 0$, then*

(i) $\beta_{i,j}(I) = \beta_{i,j}(J) + \beta_{i-1,j-1}(J) + \beta_{i,j-1}(K)$,
(ii) *If $J \neq 0$, then* $\mathrm{pd}(I) = \max\{\mathrm{pd}(J) + 1, \mathrm{pd}(K)\}$,
(iii) $\mathrm{reg}(R/I) = t - 1$.

Let $c_t(G)$ be the minimum number of co-chordal subgraphs of G required to cover the t-cliques of G, i.e., the minimum number of co-chordal subgraphs so that any t-clique of G is contained in one of these subgraphs. We give an upper bound for $\mathrm{reg}(R/K_t(G))$ in terms of $c_t(G)$.

Theorem 2.3. *Let G be a graph. Then* $\mathrm{reg}(R/K_t(G)) \leq (t-1)c_t(G)$.

Theorem 2.4. *Let n and t be positive integers. Then $\Delta_{K_t(P_n^c)}$ and $\Delta_{K_t(C_n^c)}$ are pure shellable and hence $R/K_t(P_n^c)$ and $R/K_t(C_n^c)$ are Cohen-Macaulay.*

For a graph G, we define the t-independence ideal of G as $J_t(G) = \bigcap_{\{x_{i_1},\dots,x_{i_t}\}\in\Delta_G} \langle x_{i_1}, \dots, x_{i_t}\rangle$. Indeed $J_t(G) = K_t(G^c)^\vee$.

Theorem 2.5. *Let G be a chordal graph. Then*

(i) *$\Delta_{J_t(G)}$ is pure vertex decomposable.*
(ii) *$R/J_t(G)$ is Cohen-Macaulay.*
(iii) *If $J_t(G) \neq 0$, then* $\mathrm{pd}(R/J_t(G)) = t$.
(iv) *If u is a simplicial vertex of G and $J_t(G\setminus u) \neq 0$, then* $\mathrm{reg}(R/J_t(G)) = \max\{\mathrm{reg}(R/J_t(G \setminus u)) + 1, \mathrm{reg}(R/J_{t-1}(G \setminus N_G[u]))\}$.

Corollary 2.6. *Let n and t be positive integers. Then $J_t(P_n)$ and $J_t(C_n)$ have linear quotients and hence a $(n-2t+2)$-linear resolution if they are nonzero ideals.*

Corollary 2.7. *Let n and t be positive integers with $n \geq 2t-1$. Then* $\mathrm{pd}(K_t(P_n^c)) = n - 2t + 1$. *In particular, for $n \geq 3$,* $\mathrm{pd}(I(P_n^c)) = n - 3$.

Corollary 2.8. *Let n and t be positive integers such that $n \geq 2t$. Then* $\mathrm{pd}(K_t(C_n^c)) = n - 2t + 1$. *In particular* $\mathrm{pd}(I(C_n^c)) = n - 3$.

Theorem 2.9. *Let n and t be positive integers such that $n \geq 2t-1$. Then*

$$\beta_{i,j}(J_t(P_n)) = \beta_{i,j-1}(J_t(P_{n-1})) + \beta_{i,j}(J_{t-1}(P_{n-2})) + \beta_{i-1,j-1}(J_{t-1}(P_{n-2})).$$

Theorem 2.10. *Let n and t be positive integers such that $n \geq 2t$. Then* $\operatorname{pd}(R/J_t(C_n)) = 2t - 1$.

The next result determines $\operatorname{Ass}(I^k)$ for the ideal $I = K_t(P_n^c)$ for arbitrary t, n and k.

Theorem 2.11. *Let $n, t > 1$ and k be positive integers such that $n \geq 2t - 1$ and $I = K_t(P_n^c)$.*

(i) *If $n = 2t - 1$, then* $\operatorname{Ass}(I^k) = \operatorname{Ass}(I) = \{\langle x_1\rangle, \langle x_3\rangle, \ldots, \langle x_{2t-1}\rangle\}$.

(ii) *If $n = 2t$, then*

$$\operatorname{Ass}(I^k) = \operatorname{Ass}(I) = \{\langle x_{i_1}, x_{i_2}\rangle : \ i_1 < i_2, \ i_1 \textit{ is odd and } i_2 \textit{ is even}\}.$$

(iii) *If $n > 2t$, then*

$$\operatorname{Ass}(I^k) = \{\langle x_{i_1}, x_{i_2}, \ldots, x_{i_{n-2t+2\ell}}\rangle : \ i_1 < i_2 < \cdots < i_{n-2t+2\ell},$$
$$1 \leq \ell \leq \min\{t, k\}, \ \forall\, 1 \leq j \leq n-2t+2\ell, \ i_j \textit{ and } j \textit{ have the same parity}\}.$$

Corollary 2.12. *Let n and $t > 1$ be positive integers such that $n \geq 2t - 1$. Then $I = K_t(P_n^c)$ has the persistence property. Also*

(i) *If $n = 2t - 1$ or $n = 2t$, then $astab(I) = 1$, hence I is normally torsion-free and* $\operatorname{Ass}^\infty(I) = \operatorname{Ass}(I)$.

(ii) *If $n > 2t$, then $astab(I) = t$ and*

$$\operatorname{Ass}^\infty(I) = \{\langle x_{i_1}, x_{i_2}, \ldots, x_{i_{n-2t+2\ell}}\rangle : \ i_1 < i_2 < \cdots < i_{n-2t+2\ell},$$
$$1 \leq \ell \leq t, \forall\, 1 \leq j \leq n - 2t + 2\ell, \ i_j \textit{ and } j \textit{ have the same parity}\}.$$

References

1. Moradi, S.: t-clique ideal and t-independence ideal of a graph. Communications in Algebra, **46**, 3377–3387 (2018).
2. Moradi, S., Khosh-Ahang, F.: On vetex decomposable simplicial complexes and their Alexander duals. Math. Scand., **118**, 43–56 (2016).
3. Moradi, S., Rahimbeigi, M., Khosh-Ahang, F., Soleyman Jahan, A.: A family of monomial ideals with the persistence property. To appear in J. Algebra Appl., (2018).
4. Villarreal, R. H.: Cohen-Macaulay graphs. Manuscripta Math. , **66**, 277–293 (1990).