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# ON THE CHARACER DEGREE GRAPH OF FINITE GROUPS

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Abstract. Let $G$ be a finite group, and let $\Delta(G)$ denote the *prime graph* built on the set of degrees of the irreducible complex characters of $G$. A fundamental result by P.P. Pálfy asserts that the complement $\bar{\Delta}(G)$ of the graph $\Delta(G)$ does not contain any cycle of length 3. In this paper we generalize Pálfy's result, showing that $\bar{\Delta}(G)$ does not contain any cycle of odd length, if $G$ dose note have any normal subgroup isomorphic to a projective special linear group or an special linear group of degree 2. So if $G$ does not have such a normal subgroup, then it is a bipartite graph. As an immediate consequence, the set of vertices of $\Delta(G)$ can be covered by two subsets, each inducing a complete subgraph. The latter property yields in turn that if $n$ is the clique number of $\Delta(G)$, then $\Delta(G)$ has at most $2n$ vertices. This confirms a conjecture by Z. Akhlaghi and H.P. Tong-Viet. In the case $G$ has a normal subgroup isomorphic to a projective special linear group or an special linear group of degree 2, then $\Delta(G)$ has at most $3n$ vertices, where $n$ is the clique number of $\Delta(G)$. These results provides some evidence for the famous $\rho$-$\sigma$ *conjecture* by B. Huppert.

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## 1. Introduction

Character Theory is one of the fundamental tools in the theory of finite groups, and, given a finite group $G$, the study of the set $\mathrm{cd}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$, of all degrees of the irreducible complex characters of $G$, is a particularly intriguing aspect of this theory. One of the methods that have been devised to approach such degree-set is to consider the *prime graph* $\Delta(G)$ attached to it.

The character degree graph $\Delta(G)$ is thus defined as the (simple undirected) graph whose vertex set is the set $\mathrm{V}(G)$ of all the prime numbers that divide some $\chi(1) \in \mathrm{cd}(G)$, while a pair $\{p, q\}$ of distinct vertices $p$ and $q$, belongs to the edge set $\mathrm{E}(G)$ if and only if $pq$ divides an element in $\mathrm{cd}(G)$.

As regards the investigation about general properties of $\Delta(G)$, the celebrated Ito-Michler Theorem may be regarded as the first crucial step: this fundamental result characterizes $\mathrm{V}(G)$ as the set of all primes $p$ for which $G$ does not have an abelian normal Sylow $p$-subgroup.

On the other hand, another fundamental result *in the context of solvable groups* is Pálfy's "Three-Vertex Theorem": given any three distinct primes in $\mathrm{V}(G)$, at least two of them are adjacent in $\Delta(G)$. For instance, the bound of 3 for the diameter in the connected case, as well as the structure of the non-connected case as the union of two complete subgraphs, are straightforward consequences of Pálfy's theorem. A theorem that may be rephrased by saying that, for a finite solvable group $G$, the complement of $\Delta(G)$ (that we will denote by $\bar{\Delta}(G)$) does not contain any triangle.

We extend Palfy's theorem. We stress that Palfy's theorem is actually proved here with an argument which embeds naturally into our analysis (specifically, in the last two paragraphs of the proof of Theorem A). This makes our treatment essentially self-contained.

**Theorem A.** Let $G$ be a group, and let $\pi \subseteq \mathrm{V}(G)$ be such that $|\pi|$ is an odd number larger than 1. Then $\pi$ is the set of vertices of a cycle in $\overline{\Delta}(G)$ if and only if $\mathbf{O}^{\pi'}(G) = S \times A$, where $A$ is abelian, $S \simeq SL_2(u^\alpha)$ or $S \simeq PSL_2(u^\alpha)$ for a prime $u \in \pi$ and a positive integer $\alpha$, and the primes in $\pi \setminus \{u\}$ are alternately odd divisors of $u^\alpha + 1$ and $u^\alpha - 1$.

An immediate consequence is the following.

**Corollary B.** *Let $G$ be a finite group and assume that $G$ does not have any normal subgroup isomorphic to a projective special linear group or an special linear group of degree* 2*. Then $\mathrm{V}(G)$ is covered by two subsets, each inducing a clique (i.e. a complete subgraph) in $\Delta(G)$.*

*In particular, for every subset* $\mathcal{S}$ *of* $\mathrm{V}(G)$, *at least half of the vertices in* $\mathcal{S}$ *are pairwise adjacent in* $\Delta(G)$.

The fact that, for a finite solvable group $G$, the set $\mathrm{V}(G)$ is covered by two subsets each inducing a clique was already known to be true in two special cases: when $\Delta(G)$ is disconnected (as already mentioned), and, in the connected case, when the diameter of $\Delta(G)$ attains the upper bound 3. By Corollary B, this is indeed a feature of $\Delta(G)$ in full generality.

Again by Corollary B, if $G$ is a finite group describing in the Corollary B, and $n$ is the maximum size of a clique in $\Delta(G)$, then $|\mathrm{V}(G)| \leq 2n$. This is precisely what is conjectured by Akhlaghi and Tong-Viet in [3] for solvable groups.

Moreover, since the distinct prime divisors of a single irreducible character degree do of course induce a clique in $\Delta(G)$, Corollary B provides some more evidence for the famous (and still open) *$\rho$-$\sigma$ conjecture* by Huppert, which predicts that any finite solvable group $G$ has an irreducible character whose degree is divisible by at least half the primes in $\mathrm{V}(G)$.

Finally we just recall that, as shown for instance by $\mathrm{Alt}(5)$, our theorem (as well as Pálfy's) does not hold for the groups having a normal subgroup isomorphic to a projective special linear group or an special linear group of degree 2.

## References

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